Rotations in SO(N)

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A plane can be designated as the exterior product of any two non-parallel vectors that are contained within it. For example the plane that contains the X and Y axis in a 3 dimensional Euclidean Space would be represented as:

$$\vec{x} \wedge \vec{y} \neq 0$$

In order to parametrize a rotation in N-dimensional Euclidean space, around a point, we must first define a set of fixed planes in our space around which rotations will occur; the quantity of planes needed to represent all possible rotations within our space is equivalent to the number of parameters of the Special Orthogonal Group of order N, SO(N). Each plane must be orthogonal (perpendicular) to the others, and the minimum number of such planes is determined from the dimensionality of our Euclidean Space. For simplicity each plane of rotation will be one of the planes defined by any pair of standard basis vectors for our Euclidean Space, all of which will pass through the origin, although any set of orthogonal vectors which span the entire Euclidean Space could have been used. In 3 dimensional Euclidean Space where we have the 3 dimensions, x, y, and z. The rotation of this space is the 3-parameter Lie Group, SO(3), requiring 3 planes of rotation: $\vec{x} \wedge \vec{y}$, $\vec{x} \wedge \vec{z}$, and $\vec{y} \wedge \vec{z}$. Similarly, in 4 dimensional Euclidean Space we have the 4 dimensions, x, y, z, and w. The rotation of this space is the 6-parameter Lie group, SO(4), requiring 6 planes of rotation: $\vec{x} \wedge \vec{y}$, $\vec{x} \wedge \vec{z}$, $\vec{x} \wedge \vec{w}$, $\vec{y} \wedge \vec{z}$, $\vec{y} \wedge \vec{w}$, and $\vec{z} \wedge \vec{w}$. The number of planes of rotation required for a specific number of dimensions can be calculated as the triangle number of N-1, where N is the number of dimensions of our Euclidean Space.

$$T_{N-1} = \frac{N(N-1)}{2} = 1 + 2 + 3 + 4 + ...(N-1)$$

Another way to think about it is that the set of the planes of rotation are all the 2-element subsets from the set of all the standard basis of our Euclidean Space.

$$P = \{p \,|\, p \in \wp(E), |p| = 2\}$$

Where P is a set of planes of rotation for our Euclidean Space. E is the set of all the standard basis within our Euclidean Space, $\wp(E)$ is the power set of E, and |p| is the cardinality of subset p. P is essentially the set of all 2-element subsets of E.

Since the quantity of standard basis within a Euclidean Space is always equal to the number of dimensions, the cardinality of P will always be equal to the binomial coefficient N over 2 and the triangle number of N-1.

$$|P| = \binom{N}{2} = T_{N-1}$$

However we will want to represent P as a sequence of elements, instead of as a set of elements. This is because the order in which we apply each rotation around each plane will matter, therefore we must provide an ordering to each plane of rotation. Suppose now that E is a sequence of all the standard basis of our Euclidean Space, then we construct the sequence P as follows:

$$E = (e_1, e_2, e_3, ...e_N)$$

$$P = \left(\left(p_{T_{N-1} - T_{N-n} + m} \right)_{m=n+1}^{N} \right)_{n=1}^{(N-1)}, e_n \wedge e_m$$

If we wish to define a rotation about a plane that contains two of the standard basis of our Euclidean Space, $\vec{x} \wedge \vec{y}$, we can construct a rotation matrix, R, as follows:

$$\vec{x} \wedge \vec{y} \in P$$

$$R(\vec{x} \wedge \vec{y}, \theta) = \Re((sin(\theta) - j * cos(\theta) + j) * ((\check{X} + j * \check{Y})^T \cdot (\check{Y} + j * \check{X}))) + I$$

Where $\check{X}$ is the row vector matrix of $\vec{x}$, $\check{Y}$ is the row vector matrix of $\vec{y}$, I is the NxN identity matrix, j is the imaginary number, and $\Re$ is the function for obtaining the real part. Also $*$ is the operator for scalar multiplication and $\cdot$ is used to represent the matrix multiplication.

For example if:

$$\check{X} = [1, 0, 0, 0...]$$

$$\check{Y} = [0, 1, 0, 0...]$$

Then:

$$\check{X} + j * \check{Y} = [1, j, 0, 0...]$$

$$\check{Y} + j * \check{X} = [j, 1, 0, 0...]$$

$$(\check{X} + j * \check{Y})^T \cdot (\check{Y} + j * \check{X}) = \begin{bmatrix} j & 1 & 0 & 0 & \cdots \\ -1 & j & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{bmatrix}$$

$$R = \begin{bmatrix} cos(\theta) & sin(\theta) & 0 & 0 & \cdots \\ -sin(\theta) & cos(\theta) & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & \cdots \\ 0 & 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{bmatrix}$$

Function R also has the following domain and range:

$$R : (\mathbb{R}^N \wedge \mathbb{R}^N x \mathbb{R}) \rightarrow \mathbb{R}^{NxN}$$

Such that both input vectors must have the same dimension, N, and the output matrix will be a square matrix with dimension NxN.

To perform a rotation using a rotation matrix calculate the dot product between an untransformed row vector matrix and the rotation matrix.

$$\check{B} = \check{A} \cdot R$$

Where $\check{A}$ is the untransformed row vector matrix, R is the rotation matrix, and $\check{B}$ is the rotated row vector matrix.

All Three matrices $\check{A}$, $\check{B}$, and R must agree on the dimensionality, N, such that:

$$\mathbb{R}^N \cdot \mathbb{R}^{NxN} \rightarrow \mathbb{R}^N$$

However, as stated earlier, in order to parametrize all possible rotations on vectors within our Euclidean Space we must combine multiple planes of rotation at once; one

for every pair of standard basis within our space. Therefore for every plane of rotation p_r in P, there will be an associated angle, θ_r in our corresponding sequence of rotation angles Θ. Such that to do a rotation about all the planes of rotation we do:

$$P = (p_1, p_2, p_3, ...p_{T_{N-1}})$$

$$\Theta = (\theta_1, \theta_2, \theta_3, ...\theta_{T_{N-1}})$$

$$\check{v}_r = \check{v} \cdot R(p_1, \theta_1) \cdot R(p_2, \theta_2) \cdot R(p_3, \theta_3) \cdot ... R(p_{T_{N-1}}, \theta_{T_{N-1}})$$

$$\check{v}_r = \check{v} \cdot \prod_{r=1}^{T_{N-1}} R(p_r, \theta_r)$$

Where the product operator (uppercase pi, $\prod$) represents matrix multiplication instead of scalar multiplication, $\check{v}_r$ is our rotated row vector matrix, $\check{v}$ is our untransformed row vector matrix, and the R function is the function for constructing our rotation matrix from earlier.

From this we can define our rotational transformation in N-dimensions as the following function:

$$Rot(\vec{v}, \vec{\theta})$$

Such that:

$$Rot : (\mathbb{R}^N x \mathbb{R}^{T_{N-1}}) \rightarrow \mathbb{R}^N$$

That is, it takes a vector of dimension N, representing our untransformed vector; as well as a vector of dimension triangle number N-1 representing the angle of rotation around each plane of rotation. Our function then returns a rotated vector of the same dimensionality as our untransformed vector.