Demonstrations for Modeling

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Part I

Fundamentals

CHAPTER 1

PHASORS

Phasors are a way of representing time varying signals as complex numbers in the frequency domain. That is, values in the form $a + bj$ where j represents the imaginary number. Phasors, and complex numbers in general can also be represented in polar form as $A\angle\theta$. One can imagine the vector plotted on a plane where the horizontal axis represents the real part of the complex number, variable a, and the vertical axis representing the imaginary part of the complex number, variable b. This in turn would plot the vector $A\angle\theta$. as shown in figure 1.

The length of the vector, represented by variable A, is called the magnitude or modulus of the complex number and is often represented by the same symbol used for taking the absolute value, for example $|a + bj| = A$. The angle of the vector, variable θ, is called the phase or argument of the complex number. This is often represented by the function arg or argument, for example $arg(a + bj) = \theta$.

A phasor is best understood conceptually when represented in polar form.The magnitude of the phasor, A, represents the amplitude of the wave, most often the peak value however it can also represent the RMS value, though in this case the subscript rms is usually used to make this explicit. The argument of the phasor, θ, represents the phase of the signal at $t = 0$. However because, usually, the relative phase between two signals wont change over time in the more general case the phase can be seen as representing the relative phase to any other phasor in the same circuit at any given time.

Figure 2 shows some examples of voltage phasors all at the same frequency however with different magnitudes and phases.

Using phasors to represent AC circuits is called Frequency-Domain Analysis. It allows us to analyze AC circuits in a way that looks sort of like DC. This usually makes the math for phasors much simpler than Time-Domain Analysis. However its important to understand that values obtained are not instantaneous values as with Time-Domain Analysis.

Another important point is the fact that the phasor itself, while representing AC signal, it does not encode the frequency of the signal. The frequency is added into the equations when calculating the impedance of the components instead.

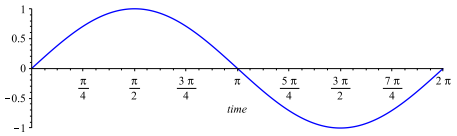

Figure 1.1: Voltage phasor for $1 + 0 * j$ in complex form or $1\angle 0$ in polar form

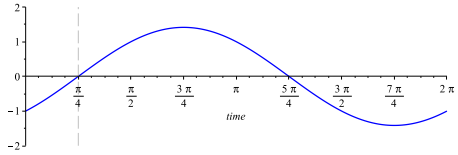

Figure 1.2: Voltage phasor for $1 + 1 * j$ in complex form or $\sqrt{2}\angle\frac{\pi}{4}$ in polar form

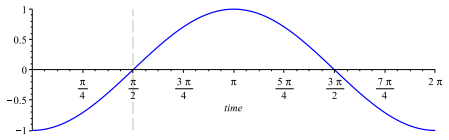

Figure 1.3: Voltage phasor for $0 + 1 * j$ in complex form or $1\angle\frac{\pi}{2}$ in polar form

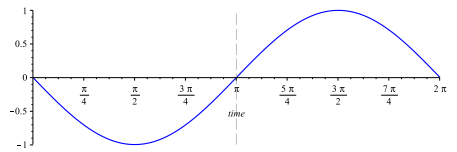

Figure 1.4: Voltage phasor for $-1 + 0 * j$ in complex form or $1\angle\pi$ in polar form

CHAPTER 2

IMPEDANCE

Impedance represents the other piece of the puzzle for frequency-domain analysis. Impedance is also a complex number that represents the opposition a circuit presents to the flow of current. Similar to how any network of resistors can be reduced to a single resistor (thevenin's equivalent) a single impedance value can represent any combination of resistors, inductors, and capacitors.

Impedance is defined as the complex ratio o the voltage phasor and the crrnt phasor.

$$Z = \frac{V}{I} \tag{2.0.1}$$

Impedance is a complex number, and like all complex numbers consists of a real and an imaginary part in the form $a + bj$. The real value of impedance represents the resistance, this value comes from the resistive components in the circuit being represented. The imaginary value of impedance represents the reactance of the circuit. This value comes from the inductors and capacitors in the circuit where capacitors have negative values of reactance (called capacitive susceptance) and inductors have positive values of reactance (called inductive reactance). Consequently capacitors and inductors in series can partially or completely cancel each other out at a specific frequency. However because impedance values change with frequency a pair of inductor and capacitor that would cancel out at one frequency might have significant reactance at another frequency.

CHAPTER 3

FUNDAMENTAL COMPONENTS

3.1 Resistor

The three basic components, resistors, inductors, and capacitors, can be modeled by leveraging some definitions for impedance. Consider the following examples where Z is complex impedance, L is inductance, C is capacitance, and ω is the angular frequency.

Definition 3.1.1 — Instantaneous Current Through a Resistor (time-domain). Given a resistance, R, and an instantaneous voltage, V, then the instantaneous current, I will be equal to the voltage divided by the resistance:

$$I = \frac{V}{R} \tag{3.1.1}$$

Definition 3.1.2 — Impedance of a Resistor. Given a resistance, R, the complex impedance, Z, will be equal to the resistance:

$$Z = R + 0j \tag{3.1.2}$$

$$Z = R \tag{3.1.3}$$

Definition 3.1.3 — Current Phasor Through a Resistor (frequency-domain). Given a resistance, R, and a complex voltage phasor, V, then current, I, will be equal to the voltage divided by the resistance:

$$I = \frac{V}{R} \tag{3.1.4}$$

3.2 Inductor

Definition 3.2.1 — Impedance of an Inductor. Given an inductance, L, the complex impedance, Z, will be defined such that:

$$\begin{aligned} Z &= 0 + L\omega j \\ &= L\omega j \end{aligned} \tag{3.2.1}$$

3.3 Capacitor

Definition 3.3.1 — Impedance of a Capacitor. Given a capacitance, C, the complex impedance, Z, will be defined such that:

$$\begin{aligned} Z &= 0 + -C\omega j \\ &= -C\omega j \\ &= \frac{1}{C\omega j} \end{aligned} \tag{3.3.1}$$

To calculate the thevinin equivalent impedance for a series of components simply add the impedances of each component together. Consider the following examples where: Z is thevinin equivalent impedance, L is inductance, C is capacitance, R is resistance, and ω is angular frequency.

Exercise 3.1 What is the equation for the impedance of a resistor and an inductor in series?

$$Z = R + L\omega j \tag{3.3.2}$$

■

Exercise 3.2 What is the equation for the impedance of a resistor, a capacitor, and an inductor in series?

$$\begin{aligned} Z &= R + L\omega j + -C\omega j \\ &= \frac{R\omega C + CL\omega^2 j - j}{\omega C} \end{aligned} \tag{3.3.3}$$

■

CHAPTER 4

CIRCUIT LAWS

4.1 Ohm's Law

Most of the circuit laws, such as ohm's law and Kirchoff's laws, map to the frequency domain rather well. With Ohms law for example one can substitute Impedance (Z) in place of Resistance (R), voltage phasors for DC voltage, and current phasors for DC current. By doing this you will receive valid results in complex form. Problem (8) for example was just an application of ohm's law in the frequency domain. From Ohm's law we have the following.

Definition 4.1.1 — Ohm's Law. Given a complex voltage phasor, V, and a complex current phasor, I, the complex impedance, Z, will be defined such that:

$$Z = \frac{V}{I} \tag{4.1.1}$$

Similarly the magnitude of the impedance will be the ratio of the magnitudes of the voltage impedance to the current impedance.

$$|Z| = \frac{|V|}{|I|} \tag{4.1.2}$$

It also follows that the phase of the impedance represents the phase difference between the voltage across the component and the current through it.

$$arg(Z) = arg(V) - arg(I) \tag{4.1.3}$$

The consequence of this is that because a resistor has a purely real impedance the phase of its complex impedance will always be 0. This means the voltage across a resistor will always be in phase with the current through it. Similarly because inductors and capacitors have a purely imaginary impedance its phase will always be 90 degrees or $\frac{\pi}{2}$ radians ahead or behind. This means purely reactive components like inductors and capacitors will always see the voltage across them as 90 degrees out of phase with the current through them. Whether the phase shift is ahead 90 degrees or behind 90 degrees depends on if the reactance is positive or negative (In other words it is opposite for inductors and capacitors).

4.2 Joule's Law

However Joule's Law (often taught as part of Ohm's law) is somewhat different as you must use the conjugate of the current phasor.

Definition 4.2.1 — Joule's Law. Given a complex voltage phasor, V, and a complex current phasor, I, the complex power, S, will be

defined such that:

$$S = \frac{1}{2} V \bar{I} \tag{4.2.1a}$$

$$S = V_{rms} \, \overline{I_{rms}} \tag{4.2.1b}$$

4.3 Kirchoff's Current Law

4.4 Kirchoff's Voltage Law

Part II

Passive Circuits

CHAPTER 5

RC CIRCUITS

5.1 RC Low-pass Filter

We will begin with one of the simplest circuits you can model in the frequency domain, an RC Low-pass Filter. The purpose of this circuit is to act as a low-pass filter, which means that higher frequencies are attenuated to a greater extent than lower frequencies. In layman's terms it selectively allows lower frequencies to pass but not higher frequencies. What we will see is that as the frequency approaches DC then the voltage phasor at V_{out} will begin to match the phasor at V_{in}. Similarly as the frequency approaches infinity V_{out} will approach 0.

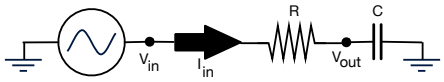

Figure 5.1: RC Low-pass Filter Schematic

Using Ohm's law we know that the voltage across the entire circuit will equal the current through it multiplied by its impedance. Since impedance is treated much like resistance we simply add the impedance of the resistor and capacitor to get the thevenin equivalent impedance of the circuit then multiply that by I_{in}.

$$V_{in} = I_{in}(Z_R + Z_C) \tag{5.1.1}$$

We can therefore solve for I_{in} and arrive at the function defining the current through the circuit.

$$I_{in} = \frac{V_{in}}{Z_R + Z_C} \tag{5.1.2}$$

Now that we know V_{in} and I_{in} we can calculate V_{out}. We know from Ohm's law that the current through the capacitor multiplied by the impedance of the capacitor will give us the voltage across the capacitor, which is V_{out}.

$$V_{out} = I_{in}\, Z_C \tag{5.1.3}$$

Next we expand our impedances with the component values.

$$Z_C = \frac{-j}{\omega C} \tag{5.1.4}$$

$$Z_R = R \tag{5.1.5}$$

Using substitution on equation (5.1.1) from earlier and then solving

for I_{in} we have the following.

$$\begin{aligned}
V_{in} &= I_{in}(R + \frac{-j}{\omega C}) \\
\frac{V_{in}}{I_{in}} &= \frac{I_{in}(R + \frac{-j}{\omega C})}{I_{in}} \\
\frac{V_{in}}{I_{in}} &= \frac{\cancel{I_{in}}(R + \frac{-j}{\omega C})}{\cancel{I_{in}}} \\
\frac{V_{in}}{I_{in}} &= R + \frac{-j}{\omega C} \\
\frac{I_{in}}{V_{in}} &= \frac{1}{R + \frac{-j}{\omega C}} \\
\frac{I_{in}}{V_{in}} V_{in} &= \frac{1}{R + \frac{-j}{\omega C}} V_{in} \\
\frac{I_{in}}{\cancel{V_{in}}} \cancel{V_{in}} &= \frac{V_{in}}{R + \frac{-j}{\omega C}} \\
I_{in} &= \frac{V_{in}}{R + \frac{-j}{\omega C}}
\end{aligned} \tag{5.1.6}$$

Lets put the equation in complex form so it is easier to work with.

$$\begin{aligned}
I_{in} &= \frac{V_{in}\omega C}{(R + \frac{-j}{\omega C})\omega C} \\
&= \frac{V_{in}\omega C}{R\omega C + \frac{-j}{\omega C}\omega C} \\
&= \frac{V_{in}\omega C}{R\omega C + \frac{-j}{\cancel{\omega C}}\cancel{\omega C}} \\
&= \frac{V_{in}\omega C}{R\omega C - j} \\
&= \frac{V_{in}\omega C(R\omega C + j)}{(R\omega C - j)(R\omega C + j)} \\
&= \frac{V_{in}R\omega^2C^2 + V_{in}\omega Cj}{R^2\omega^2C^2 + 1} \\
&= \frac{V_{in}R\omega^2C^2}{R^2\omega^2C^2 + 1} + \frac{V_{in}\omega C}{R^2\omega^2C^2 + 1}j
\end{aligned} \tag{5.1.7}$$

Similarly we can use substitution on equation (5.1.3) from earlier.

$$\begin{aligned}
V_{out} &= \frac{V_{in}}{R+\frac{-j}{\omega C}}\,\frac{-j}{\omega C} \\
&= \frac{-V_{in}j}{(R+\frac{-j}{\omega C})\omega C} \\
&= \frac{-V_{in}j}{R\omega C+\frac{-j}{\omega C}\omega C} \\
&= \frac{-V_{in}j}{R\omega C+\frac{-j}{\cancel{\omega C}}\cancel{\omega C}} \\
&= \frac{-V_{in}j}{R\omega C-j} \\
&= \frac{(-V_{in}j)(R\omega C+j)}{(R\omega C-j)(R\omega C+j)} \\
&= \frac{V_{in}-V_{in}R\omega Cj}{R^2\omega^2C^2+1} \\
&= \frac{V_{in}}{R^2\omega^2C^2+1}-\frac{V_{in}R\omega C}{R^2\omega^2C^2+1}j
\end{aligned} \tag{5.1.8}$$

Now that we have some equations to model the circuit lets try graphing a few values to get a better understanding of how the circuit behaves. First lets graph V_{out} as a function of the frequency of the signal with a constant value of 1 for both C and R. Of course, since V_{out} is a complex number we had to graph the amplitude and phase components of V_{out} separately.

We can also overlay several similar graphs to illustrate how the circuit behaves for different values of R.

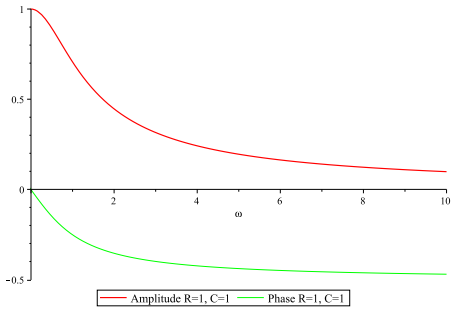

Figure 5.2: Frequency response graph for a RC circuit.

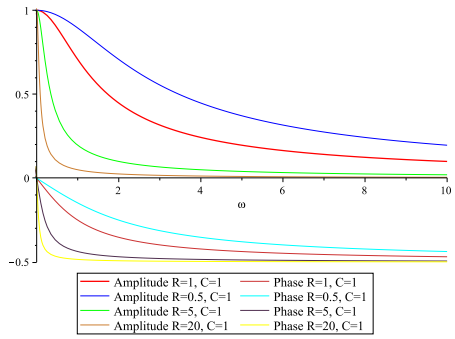

Figure 5.3: Multiple frequency response graphs for various RC circuits.

Exercise 5.1 Given the circuit in figure 5.1 where V_{in} is a sinusodial signal with a peak voltage of 12 volts, and an angular frequency of 100π. In addition the resistor, R, has a value of 10 Ohms, and the capacitor, C, has a capacitance of 2 Farads. What is the peak voltage of the signal at V_{out}?

Using equation (5.1.8) for calculating V_{out} we have:

$$\begin{aligned}
V_{out} &= \frac{V_{in}}{R^2\omega^2C^2+1}-\frac{V_{in}R\omega C}{R^2\omega^2C^2+1}j \\
&= \frac{(12+0j)}{(10)^2(100\pi)^2(2)^2+1}-\frac{(12+0j)(10)(100\pi)2}{(10)^2(100\pi)^2(2)^2+1}j \\
&= \frac{12}{4000000\pi^2+1}-\frac{24000\pi}{4000000\pi^2+1}j
\end{aligned} \tag{5.1.9}$$

■

5.2 RC High-pass Filter

CHAPTER 6

RLC CIRCUITS

6.1 Resonance

Since we know the circuit's resonant frequency is where the circuit attenuates the signal to a minimum we can easily calculate it. In other words the impedance of the circuit will be at a minimum at the resonant frequency. This occurs because for any value of capacitance and inductance there will be some frequency where the two impedances will be equal and opposite, thus they will cancel each other out leaving only the resistor. Therefore we can calculate the resonant frequency by calculating the point where the impedances are equal but opposite.

$$-Z_C = Z_L \tag{6.1.1}$$

$$\frac{j}{\lambda C} = \lambda L j \tag{6.1.2}$$

$$\lambda = \pm\frac{1}{\sqrt{LC}} \tag{6.1.3}$$

But since L and C will always be positive values what we really wind up is just the positive version.

Definition 6.1.1 — Resonant Frequency of an LC Circuit. Given an inductance, L, and a capacitance, C, the resonant frequency, λ will be as follows:

$$\lambda = \frac{1}{\sqrt{LC}} \tag{6.1.4}$$

R The resonant frequency, λ, is represented as an angular frequency in radians.

6.2 RLC Band-pass Filter

We will now apply the same principles we used in the last demonstration to a slightly more complicated demonstration of a RLC band-pass filter. A band-pass filter is a type of filter that allows frequencies to pass at specific frequency, called the resonant frequency, but will attenuate frequencies both above and below the resonant frequency. Figure 4 illustrates one configuration producing an RLC band-pass filter.

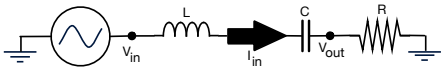

Figure 6.1: RLC Bandpass Filter Schematic

We know, due to Ohm's Law, that the voltage across the circuit will be equal to the current through the circuit multiplied by the thevenin equivalent impedance of the circuit. Furthermore we know that the thevenin equivalent impedance of the circuit will simply be the sum of the impedance of our three components. We will use this equation to arrive at a function defining the current through the circuit.

$$V_{in} = I_{in}(Z_L + Z_R + Z_C) \tag{6.2.1}$$

Solving for I_in we get

$$I_{in} = \frac{V_{in}}{Z_L + Z_R + Z_C} \tag{6.2.2}$$

Next we can calculate V_{out} using Ohm's law again. We know that the voltage across the resistor will be equal to the current through it multiplied by its impedance.

$$V_{out} = I_{in} Z_R \tag{6.2.3}$$

We now expand the impedance values with our specific component values.

$$Z_L = \omega L j \tag{6.2.4}$$

$$Z_C = \frac{-j}{\omega C} \tag{6.2.5}$$

$$Z_R = R \tag{6.2.6}$$

$$V_{in} = I_{in} * (\omega L j + \frac{-j}{\omega C} + R) \tag{6.2.7}$$

Next we solve for I_{in} and arrive at the following.

$$I_{in} = \frac{V_{in} \omega C}{R\omega C + j\omega^2 LC - j} \tag{6.2.8}$$

We can also substitute our equation for I_{in} into our equation for V_{out} arriving at a function for V_{out} that is independent of I_{in}.

$$\begin{aligned} V_{out} &= \frac{V_{in} \omega C}{R\omega C + j\omega^2 LC - j} R \\ &= \frac{V_{in} \omega C R}{R\omega C + j\omega^2 LC - j} \end{aligned} \tag{6.2.9}$$

Now that we have some equations to model the circuit lets try graphing a few values to get a better understanding of how the circuit behaves. First lets graph V_{out} as a function of the frequency of the signal with a constant value of 1 for C, L, and R. Of course, since V_{out} is a complex number we had to graph the amplitude and phase components of V_{out} separately.

We can also overlay several similar graphs to illustrate how the circuit behaves for different values of R.

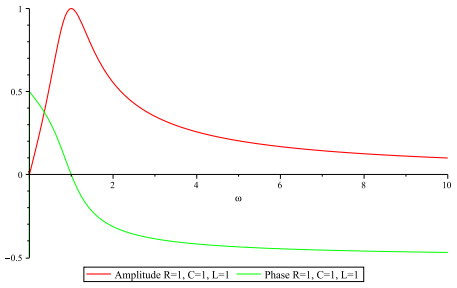

Figure 6.2: Frequency response graph for a RLC circuit.

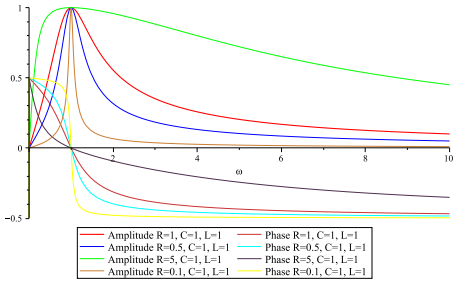

Figure 6.3: Multiple frequency response graphs for various RLC circuits.

CHAPTER 7

TRANSMISSION LINES

7.1 Transmission Line Reflections

In this demonstration we will approach some simple concepts in modeling a transmission line. A transmission line is a long length of wire used to carry an AC signal. While very short lengths of wire will not show any significant effects, longer wires approaching the wavelength of the signal will begin to exhibit more significant effects. The other property of a transmission line, other than the length, needed for many equations is called the characteristic impedance (usually labeled as Z0). The characteristic impedance represents the impedance the transmission line would exhibit if it were infinitely long. The value of the characteristic impedance is always a purely resistive value.

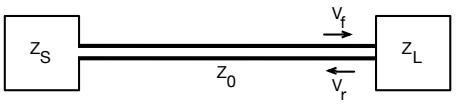

Figure 7.1: Reflections in a Transmission Line.

Typically a transmission line is terminated with an impedance on both sides of the line, and the transmission line itself represents a pair of wires, not a single wire, both carrying current in opposite directions. If a transmission line system is properly matched the source impedance, characteristic impedance, and load impedance will all have the same values. When the system is matched any waves generated at the source end will pass into the transmission line and then finally be dissipated in the load without any of the signal being reflected back towards the source. However if there is any mismatch between the impedances there will be reflections generated at the point where the mismatch occurs. In most cases only a portion of the wave is reflected; in fact the impedance needs to be either 0 or infinite in order to create a complete reflection. In addition depending on the complex impedances at the junction the reflection may also have its phase shifted.

Figure 7.1 shows a transmission line system showing a forward voltage phasor as well as the reverse voltage phasor, the reflection when the system is mismatched.

At the junction between any mismatched portions of the transmission line system we can calculate a complex value called a Reflection coefficient. The reflection coefficient defines what portion of incoming waves get reflected and to what extent it's phase will shift as a result. The reflection coefficient will have a magnitude between 0 and 1 where 0 represents no reflections and 1 indicates waves will be fully reflected.

The reflection coefficient can be defined either in terms of the impedances at the junction, or the ratio of the reverse voltage phasor to the forward voltage phasor.

Definition 7.1.1 — Reflection Coefficient from Impedance. Given a load impedance, Z_L, and a transmission line's characteristic impedance, Z_0, the reflection coefficient, Γ can be calculated as follows:

$$\Gamma = \frac{Z_L - Z_0}{Z_L + Z_0} \tag{7.1.1}$$

Definition 7.1.2 — Reflection Coefficient from Voltage Phasors. Alternatively, given a forward voltage phasor, V_f, and a reverse voltage

R SWR and Γ are both dimensionless values, as well as ratios. As such these values do not have a unit specification.

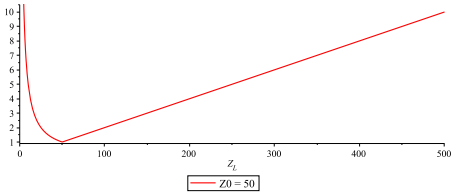

Figure 7.2: SWR plotted for a transmission line with a fixed characteristic impedance of 50 ohms with varying load impedances.

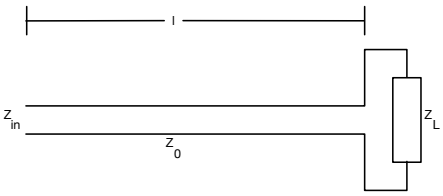

Figure 7.3: Transmission line as a transformation of a load impedance.

phasor, V_r, the reflection coefficient, Γ, can be calculated as follows:

$$\Gamma = \frac{V_r}{V_f} \tag{7.1.2}$$

Furthermore the standing wave ratio, SWR, induced int he transmission line can be calculated from the reflection coefficient.

Definition 7.1.3 — Standing Wave Ratio from Reflection Coefficient. Given a reflection coefficient, Γ, the standing wave ratio, SWR, can be calculated as follows:

$$SWR = -\frac{|\Gamma| + 1}{|\Gamma| - 1} \tag{7.1.3}$$

Next we can plot the SWR to show how it is effected by various values for the load impedance. Figure 7.2 shows how the SWR will vary given a fixed characteristic impedance on the transmission line but with various values for the load impedance. Notice when the system is matched at $Z_0 = Z_L = 50$ that the SWR is at a minimum of 1.

7.2 Impedance Transformation of a Transmission Line

Given a length of feed line terminated by an impedance we can calculate the impedance on the far end of the transmission line given the following:

$$Z_{in} = Z_0 \frac{Z_L + \tan(\beta l) Z_0 j}{Z_0 + \tan(\beta l) Z_L j} \tag{7.2.1}$$

β is defined as follows:

$$\beta = \frac{2\pi}{\lambda} \tag{7.2.2}$$

Where λ is the wavelength in the transmission line, defined as:

$$\lambda = \frac{c}{f} \tag{7.2.3}$$

Where f is the frequency in hertz, therefore to get the angular frequency, ω, we have:

$$\lambda = \frac{2\pi c}{\omega} \tag{7.2.4}$$

Finally c can be defined from the transmission line's velocity factor, VF, as follows:

$$c = VF \cdot C \tag{7.2.5}$$

Bringing it all together and we can now define our final equation.

Definition 7.2.1 — Impedance Transformation of a Transmission Line. Given a load impedance, Z_L, and the characteristic impedance of the transmission line, Z_0, and the length of the transmission line, l, and the speed of propagation through the transmission line, c, and the angular frequency of the signal, γ, then the transformed impedance

R The velocity factor, VF, is always a value between 0 and 1.

at Z_{in} is defined as follows:

$$Z_{in} = Z_0 \frac{Z_L + \tan(\frac{\omega}{c} l) Z_0 j}{Z_0 + \tan(\frac{\omega}{c} l) Z_L j} \tag{7.2.6}$$

We can now graph Z_{in} to illustrate the effects of various lengths of a transmission line. Figure 7.4 shows how the impedance is transformed for various lengths of transmission line with a characteristic impedance of 50 ohms and a load impedance of 400 ohms. Similarly figure 7.5 shows how the impedance is transformed for various lengths of transmission line with a characteristic impedance of 50 ohms and a load impedance of 10 ohms. Finally figure 7.6 shows the impedance transformation plots for an assortment of load impedances. In all of these plots the axis representing length is scaled such that a value of 1 represents a transmission line that is equal in length to the wavelength of the signal propagating through it. Furthermore all of these graphs use the angular frequency of $60,000\pi$.

It is interesting to note that starting at one-eigth of a wavelength, and repeated every quarter of a wavelength after, the transformed impedance of the load will always match the characteristic impedance of the transmission line. So at these length intervals the system will appear to be matched. Of course this only holds true so long as the frequency remains constant.

At every half of a wavelength no transformation takes place at all, Z_{in} will just equal Z_L. This point is closely related to the node in a standing wave. The transmission line forms a standing wave due to reflections forming at the mismatched end, and this length corresponds to the position in the transmission line where a node would form in the standing wave. Similarly starting at a quarter of a wavelength and repeated every half of a wavelength there after the transformation is most exaggerated. At these points Z_{in} will have the largest difference in impedance with Z_L. These points are associated with the anti-nodes of the standing wave formed in the transmission line.

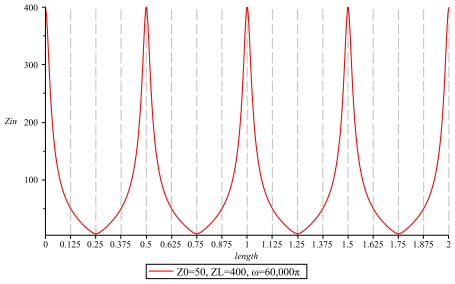

Figure 7.4: Transformed load impedance for various lengths of the transmission line.

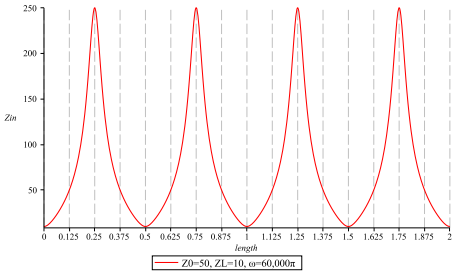

Figure 7.5: Transformed load impedance for various lengths of the transmission line.

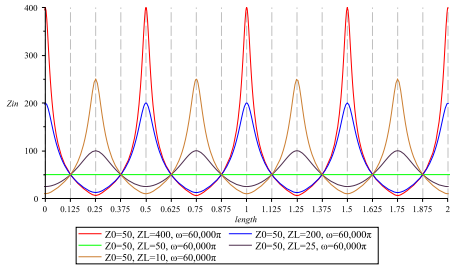

Figure 7.6: Transformed load impedance for various lengths of the transmission line.

7.3 Lumped Impedance of a Transmission Line with Load

One thing that can be useful is to reduce the transmission line itself to equivalent lumped impedances as shown in figure 7.7. The catch here is that once we actually calculate the functions for the lumped components for the transmission line we find that their values are not only dependant on the characteristic impedance and length of the transmission line but they are also a function of the frequency and the load impedance. This is a particularly useful technique when analyzing the complex power of circuits involving a transmission line as well as applying circuit laws in general.

We will begin with the equation from the last demonstration depicting Z_{in} for a transmission line as shown in the previous section.

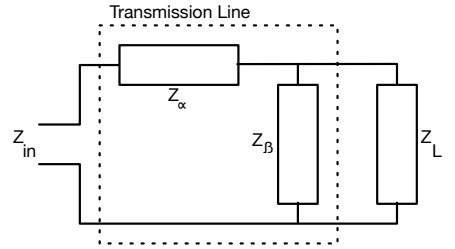

Figure 7.7: Transmission line as a lumped impedance.

CHAPTER 8

TRANSFORMERS

8.1 Transformer Coupled Circuit

Transformers are modeled similar to inductors however there are some unique consequences of the fact that their magnetic flux overlap and thus some of their inductance is mutual, labeled M in Figure 8.1. In order to model a transformer we must insert 2 virtual inductors, one in series with L_1 the other in series with L_2. However the virtual inductor is modeled such that its voltage is the inverse of the mutual impedance multiplied by the current through the coil on the opposite side of the transformer. In other words the virtual inductor added in series to L_1 would have the voltage $-Z_M I_2$ and the virtual inductor added in series to L2 would have the voltage $-Z_M I_1$.

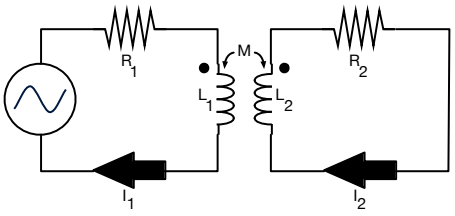

Figure 8.1: Transformer based circuit schematic.

The first step is calculating the mutual inductance of the coils in the inductor. Usually the coupling coefficient is supplied, k, which is a value between 0 and 1, with 1 representing a perfectly coupled transformer. M is calculated as follows:

$$M = k\sqrt{L_1 L_2} \tag{8.1.1}$$

To calculate the impedance representing the mutual inductance of the transformer:

$$Z_M = \omega M j \tag{8.1.2}$$

To calculate the impedance of inductor L1 or L2:

$$Z_L = \omega L j \tag{8.1.3}$$

Since we have two separate meshes (circuit loops) we will produce two separate simultaneous equations for each mesh and then use substitution. Since the sum of the voltage drops around a mesh will always be 0, we use this to construct our two simultaneous equations. The first equation is the sum of the voltage drops around the first mesh, including our virtual inductor representing the mutual inductance of the transformer.

$$0 = -V + V_{R1} + V_{L1} - V_{M1} \tag{8.1.4}$$

Substitute in Ohm's Law for voltage

$$0 = -V + (Z_{R1} I_1) + (Z_{L1} I_1) - (Z_M I_2) \tag{8.1.5}$$

Substitute values for impedance.

$$\begin{aligned} 0 &= -V + (R_1 I_1) + ((\omega L_1 j) I_1) - ((\omega M j) I_2) \\ &= -V + R_1 I_1 + \omega L_1 I_1 j - \omega M I_2 j \end{aligned} \tag{8.1.6}$$

Now we repeat the same process for mesh 2.

$$\begin{aligned} 0 &= -V + (Z_{R2}I_2) + (Z_{L2}I_2) - (Z_M I_1) \\ &= -V + (R_2 I_2) + ((\omega L_2 j)I_2) - ((\omega M j)I_1) \\ &= -V + R_2 I_2 + \omega L_2 I_2 j - \omega M I_1 j \end{aligned} \tag{8.1.7}$$

This leaves us with two simultaneous equations. We can therefore solve for I in either equation and use substitution to arrive at two independent equations. Therefore the next step is to solve for I_1 in equation 8.1.6.

$$I_1 = \frac{V + \omega M I_2 j}{R_1 + \omega L_1 j} \tag{8.1.8}$$

Now we substitute the value of I_1 into equation 8.1.7 to arrive at the independent equation for mesh 2.

$$0 = -V + R_2 I_2 + \omega L_2 I_2 j - \omega M \left(\frac{V + \omega M I_2 j}{R_1 + \omega L_1 j}\right) j \tag{8.1.9}$$

Solving for I_2 we get the following equation.

$$I_2 = \frac{\omega M V j}{R_1 R_2 + \omega L_1 R_2 j + \omega R_1 L_2 j - L_1 L_2 \omega^2 + \omega^2 M^2} \tag{8.1.10}$$

Likewise we substitute the value for I_2 from the previous equation into equation 8.1.8 to arrive at the independent equation for mesh 1.

$$I_1 = \frac{V + \omega M \left(\frac{\omega M V j}{R_1 R_2 + \omega L_1 R_2 j + \omega R_1 L_2 j - L_1 L_2 \omega^2 + \omega^2 M^2}\right) j}{R_1 + \omega L_1 j} \tag{8.1.11}$$

$$= \frac{V(R_1 R_2 + \omega L_1 R_2 j + \omega R_1 L_2 j - L_1 L_2 \omega^2)}{(R_1 R_2 + \omega L_1 R_2 j + \omega R_1 L_2 j - L_1 L_2 \omega^2 + \omega^2 M^2)(R_1 + \omega L_1 j)} \tag{8.1.12}$$

8.2 Impedance Transformation of a Transformer

CHAPTER 9

IMPEDANCE MATCHING NETWORKS

9.1 L Network Conjugate Matching

The goal of an L Network is to convert an impedance at the load to match some desired impedance. But in this demonstration we are specifically interested in an L Network which produces a conjugate match, the purpose of which is essentially that of a tuner. In other words the impedance looking towards the load will match the source impedance, and the impedance looking towards the source will equal the Complex Conjugate of the load impedance. This will allow any waves moving forward to travel normally while all waves traveling in reverse to be reflected back towards the load. This ensures more of the power reaches the load rather than being dissipated in the source.

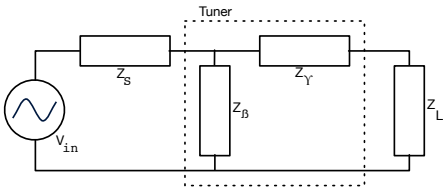

Figure 9.1: An L Conjugate Match Tuner.

First lets calculate the thevenin equivalent impedance for Z_f. To do that we first sum Z_γ and Z_L because these two elements are in series. We then take the reciprocal of that value and sum it with the recipricol of Z_β since those elements are in parallel with eachother; this will give us Z_f.

$$Z_f = \frac{1}{Z_\gamma + Z_L} + \frac{1}{Z_\beta} \tag{9.1.1}$$

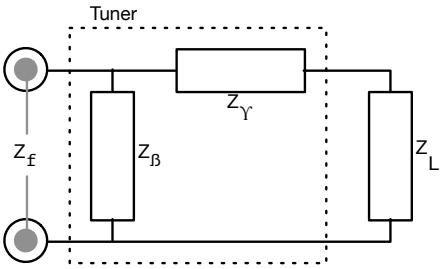

Figure 9.2: An L Conjugate Match Tuner forward looking impedance.

Similarly lets calculate the thevenin equivelant impedance for Z_r. In this case we will want to sum the recipricols of Z_β and Z_S, then sum that value with the impedance for Z_γ to arrive at Z_r.

$$Z_r = \frac{1}{Z_S} + \frac{1}{Z_\beta} + Z_\gamma \tag{9.1.2}$$

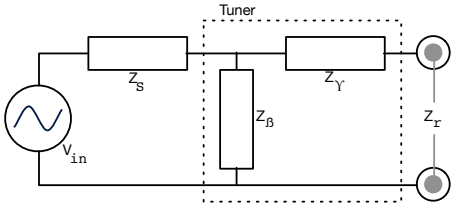

Figure 9.3: An L Conjugate Match Tuner reverse looking impedance.

As stated earlier we want Z_f to be equal to Z_S. Therefore we can take equation 9.1.1 and substitute in Z_S for Z_f.

$$Z_S = \frac{1}{Z_\gamma + Z_L} + \frac{1}{Z_\beta} \tag{9.1.3}$$

Similarly we want Z_r to be the conjugate of Z_L. Therefore we can take equation 9.1.2 and substitute in the conjugate of Z_L.

$$\overline{Z_L} = \frac{1}{Z_S} + \frac{1}{Z_\beta} + Z_\gamma \tag{9.1.4}$$

Since we now have two simultanious equations, as well as two unknown impedances, we can use substitution and arrive at two independent equations. First lets take equation 9.1.3 and solve for Z_β.

$$
\begin{aligned}
Z_S &= \frac{1}{Z_\gamma + Z_L} + \frac{1}{Z_\beta} \\
Z_S - \frac{1}{Z_\beta} &= \frac{1}{Z_\gamma + Z_L} + \frac{1}{Z_\beta} - \frac{1}{Z_\beta} \\
Z_S - \frac{1}{Z_\beta} &= \frac{1}{Z_\gamma + Z_L} + \cancel{\frac{1}{Z_\beta}} - \cancel{\frac{1}{Z_\beta}} \\
Z_S - \frac{1}{Z_\beta} &= \frac{1}{Z_\gamma + Z_L} \\
-\frac{1}{Z_\beta} + Z_S &= \frac{1}{Z_\gamma + Z_L} \\
-\frac{1}{Z_\beta} + Z_S - Z_S &= \frac{1}{Z_\gamma + Z_L} - Z_S \\
-\frac{1}{Z_\beta} + \cancel{Z_S} - \cancel{Z_S} &= \frac{1}{Z_\gamma + Z_L} - Z_S \\
-\frac{1}{Z_\beta} &= \frac{1}{Z_\gamma + Z_L} - Z_S \\
-\frac{1}{Z_\beta}(-1) &= (\frac{1}{Z_\gamma + Z_L} - Z_S)(-1) \\
\cancel{\frac{1}{Z_\beta}(-1)} &= (\frac{1}{Z_\gamma + Z_L} - Z_S)(-1) \\
\frac{1}{Z_\beta} &= \frac{-1}{Z_\gamma + Z_L} + Z_S \\
\frac{1}{Z_\beta} &= \frac{-1}{Z_\gamma + Z_L} + \frac{Z_S(Z_\gamma + Z_L)}{(Z_\gamma + Z_L)} \\
\frac{1}{Z_\beta} &= \frac{-1}{Z_\gamma + Z_L} + \frac{Z_\gamma Z_S + Z_L Z_S}{Z_\gamma + Z_L} \\
\frac{1}{Z_\beta} &= \frac{Z_\gamma Z_S + Z_L Z_S - 1}{Z_\gamma + Z_L} \\
(\frac{1}{Z_\beta})^{(-1)} &= (\frac{Z_\gamma Z_S + Z_L Z_S - 1}{Z_\gamma + Z_L})^{(-1)} \\
Z_\beta &= \frac{Z_\gamma + Z_L}{Z_\gamma Z_S + Z_L Z_S - 1}
\end{aligned}
\tag{9.1.5}
$$

Similarly we will now solve for Z_γ in equation 9.1.4.

$$\overline{Z_L} = \frac{1}{Z_S} + \frac{1}{Z_\beta} + Z_\gamma$$

$$\overline{Z_L} - \boldsymbol{Z_\gamma} = \frac{1}{Z_S} + \frac{1}{Z_\beta} + Z_\gamma - \boldsymbol{Z_\gamma}$$

$$\overline{Z_L} - Z_\gamma = \frac{1}{Z_S} + \frac{1}{Z_\beta} + Z_\gamma - Z_\gamma$$

$$-Z_\gamma + \overline{Z_L} = \frac{1}{Z_S} + \frac{1}{Z_\beta}$$

$$-Z_\gamma + \overline{Z_L} - \boldsymbol{\overline{Z_L}} = \frac{1}{Z_S} + \frac{1}{Z_\beta} - \boldsymbol{\overline{Z_L}}$$

$$-Z_\gamma + \overline{Z_L} - \overline{Z_L} = \frac{1}{Z_S} + \frac{1}{Z_\beta} - \overline{Z_L}$$

$$-Z_\gamma = \frac{1}{Z_S} + \frac{1}{Z_\beta} - \overline{Z_L}$$

$$(-Z_\gamma)(\boldsymbol{-1}) = (\frac{1}{Z_S} + \frac{1}{Z_\beta} - \overline{Z_L})(\boldsymbol{-1})$$

$$(-Z_\gamma)(-1) = (\frac{1}{Z_S} + \frac{1}{Z_\beta} - \overline{Z_L})(-1)$$

$$Z_\gamma = \frac{-1}{Z_S} - \frac{1}{Z_\beta} + \overline{Z_L} \tag{9.1.6}$$

The final step is substitution, so now we will substitute Z_γ from equation 9.1.6 into equation 9.1.5.

$$\begin{aligned}
Z_\beta &= \frac{Z_\gamma + Z_L}{Z_\gamma Z_S + Z_L Z_S - 1} \\
&= \frac{(\frac{-1}{Z_S} - \frac{1}{Z_\beta} + \overline{Z_L}) + Z_L}{(\frac{-1}{Z_S} - \frac{1}{Z_\beta} + \overline{Z_L})Z_S + Z_L Z_S - 1} \\
&= \frac{\frac{-1}{Z_S} - \frac{1}{Z_\beta} + \overline{Z_L} + Z_L}{\frac{-Z_S}{Z_S} - \frac{Z_S}{Z_\beta} + \overline{Z_L} Z_S + Z_L Z_S - 1} \\
&= \frac{\frac{-1}{Z_S} - \frac{1}{Z_\beta} + \overline{Z_L} + Z_L}{-1 - \frac{Z_S}{Z_\beta} + \overline{Z_L} Z_S + Z_L Z_S - 1} \\
&= \frac{\frac{-1}{Z_S} - \frac{1}{Z_\beta} + \overline{Z_L} + Z_L}{\frac{-Z_S}{Z_\beta} + \overline{Z_L} Z_S + Z_L Z_S - 1 - 1} \\
&= \frac{\frac{-1}{Z_S} - \frac{1}{Z_\beta} + \overline{Z_L} + Z_L}{\frac{-Z_S}{Z_\beta} + \overline{Z_L} Z_S + Z_L Z_S - 2}
\end{aligned}$$

We know that a complex number added to its conjugate will give you

twice the real part only. Therefore we can simplify things a bit further.

$$
\begin{aligned}
Z_\beta &= \frac{\frac{-1}{Z_S} - \frac{1}{Z_\beta} + \overline{Z_L} + Z_L}{\frac{-Z_S}{Z_\beta} + \overline{Z_L}Z_S + Z_L Z_S - 2} \\
&= \frac{\frac{-1}{Z_S} - \frac{1}{Z_\beta} + 2\Re\mathfrak{e}(Z_L)}{\frac{-Z_S}{Z_\beta} + \overline{Z_L}Z_S + Z_L Z_S - 2} \\
&= \frac{\frac{-1}{Z_S} - \frac{1}{Z_\beta} + 2\Re\mathfrak{e}(Z_L)}{\frac{-Z_S}{Z_\beta} + \overline{Z_L}Z_S + Z_L Z_S - 2} \\
&= \frac{\frac{-1}{Z_S} - \frac{1}{Z_\beta} + 2\Re\mathfrak{e}(Z_L)}{\frac{-Z_S}{Z_\beta} + (\overline{Z_L} + Z_L)Z_S - 2} \\
&= \frac{\frac{-1}{Z_S} - \frac{1}{Z_\beta} + 2\Re\mathfrak{e}(Z_L)}{\frac{-Z_S}{Z_\beta} + 2\Re\mathfrak{e}(Z_L)Z_S - 2}
\end{aligned}
$$

Of course we still need to solve for Z_β, the first step in doing that is to get rid of the fractions.

$$
\begin{aligned}
Z_\beta &= \frac{(\frac{-1}{Z_S} - \frac{1}{Z_\beta} + 2\Re\mathfrak{e}(Z_L))Z_\beta}{(\frac{-Z_S}{Z_\beta} + 2\Re\mathfrak{e}(Z_L)Z_S - 2)Z_\beta} \\
Z_\beta &= \frac{(\frac{-1}{Z_S} - \frac{1}{Z_\beta} + 2\Re\mathfrak{e}(Z_L))Z_\beta}{(\frac{-Z_S}{Z_\beta} + 2\Re\mathfrak{e}(Z_L)Z_S - 2)Z_\beta} \\
Z_\beta &= \frac{\frac{-Z_\beta}{Z_S} - \frac{Z_\beta}{Z_\beta} + 2\Re\mathfrak{e}(Z_L)Z_\beta}{\frac{-Z_S Z_\beta}{Z_\beta} + 2\Re\mathfrak{e}(Z_L)Z_S Z_\beta - 2Z_\beta} \\
Z_\beta &= \frac{\frac{-Z_\beta}{Z_S} - \cancel{\frac{Z_\beta}{Z_\beta}} + 2\Re\mathfrak{e}(Z_L)Z_\beta}{\cancel{\frac{-Z_S Z_\beta}{Z_\beta}} + 2\Re\mathfrak{e}(Z_L)Z_S Z_\beta - 2Z_\beta} \\
Z_\beta &= \frac{\frac{-Z_\beta}{Z_S} - 1 + 2\Re\mathfrak{e}(Z_L)Z_\beta}{-Z_S + 2\Re\mathfrak{e}(Z_L)Z_S Z_\beta - 2Z_\beta}
\end{aligned}
$$

$$
Z_\beta(-Z_S+2\Re\mathfrak{e}(Z_L)Z_S Z_\beta-2Z_\beta) = \frac{(\frac{-Z_\beta}{Z_S}-1+2\Re\mathfrak{e}(Z_L)Z_\beta)(-Z_S+2\Re\mathfrak{e}(Z_L)Z_S Z_\beta-2Z_\beta)}{-Z_S+2\Re\mathfrak{e}(Z_L)Z_S Z_\beta-2Z_\beta}
$$

$$
Z_\beta(-Z_S+2\Re\mathfrak{e}(Z_L)Z_S Z_\beta-2Z_\beta) = \frac{(\frac{-Z_\beta}{Z_S}-1+2\Re\mathfrak{e}(Z_L)Z_\beta)\cancel{(-Z_S+2\Re\mathfrak{e}(Z_L)Z_S Z_\beta-2Z_\beta)}}{\cancel{-Z_S+2\Re\mathfrak{e}(Z_L)Z_S Z_\beta-2Z_\beta}}
$$

$$
Z_\beta(-Z_S + 2\Re\mathfrak{e}(Z_L)Z_S Z_\beta - 2Z_\beta) = \frac{-Z_\beta}{Z_S} - 1 + 2\Re\mathfrak{e}(Z_L)Z_\beta
$$

Now lets do some simplification and gather all the terms on the right

hand side of the equation.

$$Z_\beta(-Z_S + 2\Re\mathfrak{e}(Z_L)Z_S Z_\beta - 2Z_\beta) = \frac{-Z_\beta}{Z_S} - 1 + 2\Re\mathfrak{e}(Z_L)Z_\beta$$

$$-Z_S Z_\beta + 2\Re\mathfrak{e}(Z_L)Z_S Z_\beta Z_\beta - 2Z_\beta Z_\beta = \frac{-Z_\beta}{Z_S} - 1 + 2\Re\mathfrak{e}(Z_L)Z_\beta$$

$$-Z_S Z_\beta + 2\Re\mathfrak{e}(Z_L)Z_S {Z_\beta}^2 - 2{Z_\beta}^2 = \frac{-Z_\beta}{Z_S} - 1 + 2\Re\mathfrak{e}(Z_L)Z_\beta$$

$$-Z_S Z_\beta + 2\Re\mathfrak{e}(Z_L)Z_S {Z_\beta}^2 - 2{Z_\beta}^2 = \frac{-Z_\beta}{Z_S} + 2\Re\mathfrak{e}(Z_L)Z_\beta - 1$$

$$(-Z_S Z_\beta + 2\Re\mathfrak{e}(Z_L)Z_S {Z_\beta}^2 - 2{Z_\beta}^2) - (-Z_S Z_\beta + 2\Re\mathfrak{e}(Z_L)Z_S {Z_\beta}^2 - 2{Z_\beta}^2) = (\frac{-Z_\beta}{Z_S} + 2\Re\mathfrak{e}(Z_L)Z_\beta - 1) - (-Z_S Z_\beta + 2\Re\mathfrak{e}(Z_L)Z_S {Z_\beta}^2 - 2{Z_\beta}^2)$$

$$\cancel{(-Z_S Z_\beta + 2\Re\mathfrak{e}(Z_L)Z_S {Z_\beta}^2 - 2{Z_\beta}^2) - (-Z_S Z_\beta + 2\Re\mathfrak{e}(Z_L)Z_S {Z_\beta}^2 - 2{Z_\beta}^2)} = (\frac{-Z_\beta}{Z_S} + 2\Re\mathfrak{e}(Z_L)Z_\beta - 1) - (-Z_S Z_\beta + 2\Re\mathfrak{e}(Z_L)Z_S {Z_\beta}^2 - 2{Z_\beta}^2)$$

$$0 = (\frac{-Z_\beta}{Z_S} + 2\Re\mathfrak{e}(Z_L)Z_\beta - 1) - (-Z_S Z_\beta + 2\Re\mathfrak{e}(Z_L)Z_S {Z_\beta}^2 - 2{Z_\beta}^2)$$

The next step in solving for Z_β is to get the equation into polynomial form of $0 = aZ_\beta{}^2 + bZ_\beta + c$.

$$0 = (\frac{-Z_\beta}{Z_S} + 2\Re\mathfrak{e}(Z_L)Z_\beta - 1) - (-Z_S Z_\beta + 2\Re\mathfrak{e}(Z_L)Z_S {Z_\beta}^2 - 2{Z_\beta}^2)$$

$$0 = \frac{-1}{Z_S}Z_\beta + 2\Re\mathfrak{e}(Z_L)Z_\beta - 1 + Z_S Z_\beta - 2\Re\mathfrak{e}(Z_L)Z_S {Z_\beta}^2 + 2{Z_\beta}^2$$

$$0 = \frac{-1}{Z_S}Z_\beta + 2\Re\mathfrak{e}(Z_L)Z_\beta - 1 + Z_S Z_\beta - 2\Re\mathfrak{e}(Z_L)Z_S {Z_\beta}^2 + 2{Z_\beta}^2$$

$$0 = \frac{-1}{Z_S}Z_\beta + 2\Re\mathfrak{e}(Z_L)Z_\beta - 1 + Z_S Z_\beta - 2\Re\mathfrak{e}(Z_L)Z_S {Z_\beta}^2 + 2{Z_\beta}^2$$

$$0 = -2\Re\mathfrak{e}(Z_L)Z_S {Z_\beta}^2 + 2{Z_\beta}^2 + \frac{-1}{Z_S}Z_\beta + 2\Re\mathfrak{e}(Z_L)Z_\beta + Z_S Z_\beta - 1$$

$$0 = -2\Re\mathfrak{e}(Z_L)Z_S {Z_\beta}^2 + 2{Z_\beta}^2 + \frac{-1}{Z_S}Z_\beta + 2\Re\mathfrak{e}(Z_L)Z_\beta + Z_S Z_\beta - 1$$

$$0 = -2\Re\mathfrak{e}(Z_L)Z_S {Z_\beta}^2 + 2{Z_\beta}^2 + \frac{-1}{Z_S}Z_\beta + 2\Re\mathfrak{e}(Z_L)Z_\beta + Z_S Z_\beta - 1$$

$$0 = (-2\Re\mathfrak{e}(Z_L)Z_S + 2){Z_\beta}^2 + (\frac{-1}{Z_S} + 2\Re\mathfrak{e}(Z_L) + Z_S)Z_\beta - 1$$

Next we can use the quadratic equation to solve for Z_β, the quadratic equation takes a polynomial in the form of $0 = ax^2 + bx + c$ into the form

$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$.

$$
\begin{aligned}
Z_\beta &= \frac{-(\frac{-1}{Z_S} + 2\Re\mathfrak{e}(Z_L) + Z_S) \pm \sqrt{(\frac{-1}{Z_S}+2\Re\mathfrak{e}(Z_L)+Z_S)^2 - 4(-2\Re\mathfrak{e}(Z_L)Z_S+2)(-1)}}{2(-2\Re\mathfrak{e}(Z_L)Z_S+2)} \\
&= \frac{\frac{1}{Z_S} - 2\Re\mathfrak{e}(Z_L) - Z_S \pm \sqrt{(\frac{-1}{Z_S} + 2\Re\mathfrak{e}(Z_L) + Z_S)^2 - 8\Re\mathfrak{e}(Z_L)Z_S + 8}}{-4\Re\mathfrak{e}(Z_L)Z_S + 4} \\
&= \frac{\frac{1}{Z_S} - 2\Re\mathfrak{e}(Z_L) - Z_S \pm \sqrt{(\frac{-1}{Z_S} + 2\Re\mathfrak{e}(Z_L) + Z_S)^2 - 8\Re\mathfrak{e}(Z_L)Z_S + 8}}{-4\Re\mathfrak{e}(Z_L)Z_S + 4} \\
&= \frac{\frac{1}{Z_S} - 2\Re\mathfrak{e}(Z_L) - Z_S \pm \sqrt{(\frac{-1}{Z_S}+2\Re\mathfrak{e}(Z_L)+Z_S)(\frac{-1}{Z_S}+2\Re\mathfrak{e}(Z_L)+Z_S) - 8\Re\mathfrak{e}(Z_L)Z_S+8}}{-4\Re\mathfrak{e}(Z_L)Z_S + 4} \\
&= \frac{\frac{1}{Z_S} - 2\Re\mathfrak{e}(Z_L) - Z_S \pm \sqrt{{Z_S}^2+4\Re\mathfrak{e}(Z_L)^2-\frac{3\Re\mathfrak{e}(Z_L)}{Z_S}+\frac{1}{{Z_S}^2}+4\Re\mathfrak{e}(Z_L)Z_S-2-8\Re\mathfrak{e}(Z_L)Z_S+8}}{-4\Re\mathfrak{e}(Z_L)Z_S + 4} \\
&= \frac{\frac{1}{Z_S} - 2\Re\mathfrak{e}(Z_L) - Z_S \pm \sqrt{{Z_S}^2+4\Re\mathfrak{e}(Z_L)^2-\frac{3\Re\mathfrak{e}(Z_L)}{Z_S}+\frac{1}{{Z_S}^2}+4\Re\mathfrak{e}(Z_L)Z_S-2-8\Re\mathfrak{e}(Z_L)Z_S+8}}{-4\Re\mathfrak{e}(Z_L)Z_S + 4} \\
&= \frac{\frac{1}{Z_S} - 2\Re\mathfrak{e}(Z_L) - Z_S \pm \sqrt{{Z_S}^2+4\Re\mathfrak{e}(Z_L)^2-\frac{3\Re\mathfrak{e}(Z_L)}{Z_S}+\frac{1}{{Z_S}^{-2}}-4\Re\mathfrak{e}(Z_L)Z_S+6}}{-4\Re\mathfrak{e}(Z_L)Z_S + 4}
\end{aligned}
$$

Being unable to simplify the equation further we have now successfuly solved for the impedance of the Z_β component.

Theorem 9.1.1 Given the L network in figure 9.1 in order to produce a conjugate match it must follow that:

$$Z_\beta = \frac{\frac{1}{Z_S} - 2\Re\mathfrak{e}(Z_L) - Z_S \pm \sqrt{{Z_S}^2+4\Re\mathfrak{e}(Z_L)^2-\frac{3\Re\mathfrak{e}(Z_L)}{Z_S}+\frac{1}{{Z_S}^2}-4\Re\mathfrak{e}(Z_L)Z_S+6}}{-4\Re\mathfrak{e}(Z_L)Z_S + 4}$$

Similarly we will also substitute Z_β from equation 9.1.5 into equation

9.1.6 in order to solve for Z_γ.

$$Z_\gamma = \frac{-1}{Z_S} - \frac{1}{Z_\beta} + \overline{Z_L}$$

$$Z_\gamma = \frac{-1}{Z_S} - \frac{1}{\frac{Z_\gamma + Z_L}{Z_\gamma Z_S + Z_L Z_S - 1}} + \overline{Z_L}$$

$$Z_\gamma = \frac{-1}{Z_S} - \frac{Z_\gamma Z_S + Z_L Z_S - 1}{Z_\gamma + Z_L} + \overline{Z_L}$$

$$Z_\gamma = \frac{-1(\boldsymbol{Z_\gamma + Z_L})}{Z_S(\boldsymbol{Z_\gamma + Z_L})} - \frac{\boldsymbol{Z_S}(Z_\gamma Z_S + Z_L Z_S - 1)}{\boldsymbol{Z_S}(Z_\gamma + Z_L)} + \frac{\overline{Z_L}\boldsymbol{Z_S}(\boldsymbol{Z_\gamma + Z_L})}{\boldsymbol{Z_S}(\boldsymbol{Z_\gamma + Z_L})}$$

$$Z_\gamma = \frac{-1(Z_\gamma + Z_L)}{Z_S(Z_\gamma + Z_L)} - \frac{Z_S(Z_\gamma Z_S + Z_L Z_S - 1)}{Z_S(Z_\gamma + Z_L)} + \frac{\overline{Z_L} Z_S(Z_\gamma + Z_L)}{Z_S(Z_\gamma + Z_L)}$$

$$Z_\gamma = \frac{-Z_\gamma - Z_L}{Z_\gamma Z_S + Z_L Z_S} - \frac{Z_\gamma Z_S Z_S + Z_L Z_S Z_S - Z_S}{Z_\gamma Z_S + Z_L Z_S} + \frac{Z_\gamma \overline{Z_L} Z_S + Z_L \overline{Z_L} Z_S}{Z_\gamma Z_S + Z_L Z_S}$$

$$Z_\gamma = \frac{(-Z_\gamma - Z_L) - (Z_\gamma Z_S Z_S + Z_L Z_S Z_S - Z_S) + (Z_\gamma \overline{Z_L} Z_S + Z_L \overline{Z_L} Z_S)}{Z_\gamma Z_S + Z_L Z_S}$$

$$Z_\gamma = \frac{-Z_\gamma - Z_L - Z_\gamma {Z_S}^2 - Z_L {Z_S}^2 + Z_S + Z_\gamma \overline{Z_L} Z_S + Z_L \overline{Z_L} Z_S}{Z_\gamma Z_S + Z_L Z_S}$$

Since we know a complex number multiplied by its conjugate yields the real part squared plus the imaginary part squared we can further simplify the equation.

$$Z_\gamma = \frac{-Z_\gamma - Z_L - Z_\gamma {Z_S}^2 - Z_L {Z_S}^2 + Z_S + Z_\gamma \overline{Z_L} Z_S + Z_L \overline{Z_L} Z_S}{Z_\gamma Z_S + Z_L Z_S}$$

$$Z_\gamma = \frac{-Z_\gamma - Z_L - Z_\gamma {Z_S}^2 - Z_L {Z_S}^2 + Z_S + Z_\gamma \overline{Z_L} Z_S + (\mathfrak{Re}(Z_L)^2 + \mathfrak{Im}(Z_L)^2) Z_S}{Z_\gamma Z_S + Z_L Z_S}$$

$$Z_\gamma = \frac{-Z_\gamma - Z_L - Z_\gamma {Z_S}^2 - Z_L {Z_S}^2 + Z_S + Z_\gamma \overline{Z_L} Z_S + \mathfrak{Re}(Z_L)^2 Z_S + \mathfrak{Im}(Z_L)^2 Z_S}{Z_\gamma Z_S + Z_L Z_S}$$

Now we will cross multiple to get rid of the fractions.

$$\frac{-Z_\gamma - Z_L - Z_\gamma {Z_S}^2 - Z_L {Z_S}^2 + Z_S + Z_\gamma \overline{Z_L} Z_S + \mathfrak{Re}(Z_L)^2 Z_S + \mathfrak{Im}(Z_L)^2 Z_S}{Z_\gamma Z_S + Z_L Z_S} = Z_\gamma$$

$$-Z_\gamma - Z_L - Z_\gamma {Z_S}^2 - Z_L {Z_S}^2 + Z_S + Z_\gamma \overline{Z_L} Z_S + \mathfrak{Re}(Z_L)^2 Z_S + \mathfrak{Im}(Z_L)^2 Z_S = Z_\gamma (Z_\gamma Z_S + Z_L Z_S)$$

$$-Z_\gamma - Z_L - Z_\gamma {Z_S}^2 - Z_L {Z_S}^2 + Z_S + Z_\gamma \overline{Z_L} Z_S + \mathfrak{Re}(Z_L)^2 Z_S + \mathfrak{Im}(Z_L)^2 Z_S = Z_\gamma Z_\gamma Z_S + Z_\gamma Z_L Z_S$$

$$-Z_\gamma - Z_L - Z_\gamma {Z_S}^2 - Z_L {Z_S}^2 + Z_S + Z_\gamma \overline{Z_L} Z_S + \mathfrak{Re}(Z_L)^2 Z_S + \mathfrak{Im}(Z_L)^2 Z_S = {Z_\gamma}^2 Z_S + Z_\gamma Z_L Z_S$$

Next we will gather all the terms on the right side of the equation

and simplify.

$$
\begin{aligned}
0 &= -Z_\gamma - Z_L - Z_\gamma Z_S^2 - Z_L Z_S^2 + Z_S + Z_\gamma \overline{Z_L} Z_S + \Re\mathrm{e}(Z_L)^2 Z_S + \Im\mathrm{m}(Z_L)^2 Z_S - (Z_\gamma^2 Z_S + Z_\gamma Z_L Z_S) \\
&= -Z_\gamma - Z_L - Z_\gamma Z_S^2 - Z_L Z_S^2 + Z_S + Z_\gamma \overline{Z_L} Z_S + \Re\mathrm{e}(Z_L)^2 Z_S + \Im\mathrm{m}(Z_L)^2 Z_S - Z_\gamma^2 Z_S - Z_\gamma Z_L Z_S \\
&= -Z_\gamma - Z_L - Z_\gamma Z_S^2 - Z_L Z_S^2 + Z_S + \Re\mathrm{e}(Z_L)^2 Z_S + \Im\mathrm{m}(Z_L)^2 Z_S - Z_\gamma^2 Z_S - Z_\gamma Z_L Z_S + Z_\gamma \overline{Z_L} Z_S \\
&= -Z_\gamma - Z_L - Z_\gamma Z_S^2 - Z_L Z_S^2 + Z_S + \Re\mathrm{e}(Z_L)^2 Z_S + \Im\mathrm{m}(Z_L)^2 Z_S - Z_\gamma^2 Z_S - Z_\gamma Z_L Z_S + Z_\gamma \overline{Z_L} Z_S \\
&= -Z_\gamma - Z_L - Z_\gamma Z_S^2 - Z_L Z_S^2 + Z_S + \Re\mathrm{e}(Z_L)^2 Z_S + \Im\mathrm{m}(Z_L)^2 Z_S - Z_\gamma^2 Z_S + (-Z_L + \overline{Z_L}) Z_\gamma Z_S \\
&= -Z_\gamma - Z_L - Z_\gamma Z_S^2 - Z_L Z_S^2 + Z_S + \Re\mathrm{e}(Z_L)^2 Z_S + \Im\mathrm{m}(Z_L)^2 Z_S - Z_\gamma^2 Z_S + (2\Re\mathrm{e}(Z_L) - 2Z_L) Z_\gamma Z_S \\
&= -Z_\gamma - Z_L - Z_\gamma Z_S^2 - Z_L Z_S^2 + Z_S + \Re\mathrm{e}(Z_L)^2 Z_S + \Im\mathrm{m}(Z_L)^2 Z_S - Z_\gamma^2 Z_S + 2\Re\mathrm{e}(Z_L) Z_\gamma Z_S - 2Z_L Z_\gamma Z_S
\end{aligned}
$$

If we want to solve for Z_γ we must now get the equations into quadratic form so we can apply the quadratic equation.

$$
\begin{aligned}
0 &= -Z_\gamma - Z_L - Z_\gamma Z_S^2 - Z_L Z_S^2 + Z_S + \Re\mathrm{e}(Z_L)^2 Z_S + \Im\mathrm{m}(Z_L)^2 Z_S - Z_\gamma^2 Z_S + 2\Re\mathrm{e}(Z_L) Z_\gamma Z_S - 2Z_L Z_\gamma Z_S \\
&= -Z_\gamma - Z_L - Z_S^2 Z_\gamma - Z_L Z_S^2 + Z_S + \Re\mathrm{e}(Z_L)^2 Z_S + \Im\mathrm{m}(Z_L)^2 Z_S - Z_S Z_\gamma^2 + 2\Re\mathrm{e}(Z_L) Z_S Z_\gamma - 2Z_L Z_S Z_\gamma \\
&= -Z_\gamma - Z_L - Z_S^2 Z_\gamma - Z_L Z_S^2 + Z_S + \Re\mathrm{e}(Z_L)^2 Z_S + \Im\mathrm{m}(Z_L)^2 Z_S - Z_S Z_\gamma^2 + 2\Re\mathrm{e}(Z_L) Z_S Z_\gamma - 2Z_L Z_S Z_\gamma \\
&= -Z_\gamma - Z_L - Z_S^2 Z_\gamma - Z_L Z_S^2 + Z_S + \Re\mathrm{e}(Z_L)^2 Z_S + \Im\mathrm{m}(Z_L)^2 Z_S - Z_S Z_\gamma^2 + 2\Re\mathrm{e}(Z_L) Z_S Z_\gamma - 2Z_L Z_S Z_\gamma \\
&= -Z_S Z_\gamma^2 - Z_\gamma - Z_S^2 Z_\gamma + 2\Re\mathrm{e}(Z_L) Z_S Z_\gamma - 2Z_L Z_S Z_\gamma - Z_L - Z_L Z_S^2 + Z_S + \Re\mathrm{e}(Z_L)^2 Z_S + \Im\mathrm{m}(Z_L)^2 Z_S \\
&= -Z_S Z_\gamma^2 - Z_\gamma - Z_S^2 Z_\gamma + 2\Re\mathrm{e}(Z_L) Z_S Z_\gamma - 2Z_L Z_S Z_\gamma - Z_L - Z_L Z_S^2 + Z_S + \Re\mathrm{e}(Z_L)^2 Z_S + \Im\mathrm{m}(Z_L)^2 Z_S \\
&= -Z_S Z_\gamma^2 - Z_\gamma - Z_S^2 Z_\gamma + 2\Re\mathrm{e}(Z_L) Z_S Z_\gamma - 2Z_L Z_S Z_\gamma - Z_L - Z_L Z_S^2 + Z_S + \Re\mathrm{e}(Z_L)^2 Z_S + \Im\mathrm{m}(Z_L)^2 Z_S \\
&= (-Z_S) Z_\gamma^2 + (-1 - Z_S^2 + 2\Re\mathrm{e}(Z_L) Z_S - 2Z_L Z_S) Z_\gamma + (-Z_L - Z_L Z_S^2 + Z_S + \Re\mathrm{e}(Z_L)^2 Z_S + \Im\mathrm{m}(Z_L)^2 Z_S)
\end{aligned}
$$

Finally we can apply the quadratic equation to solve for Z_γ.

$$
\begin{aligned}
Z_\gamma &= \frac{-(-1 - Z_S^2 + 2\Re\mathrm{e}(Z_L) Z_S - 2Z_L Z_S) \pm \sqrt{(-1 - Z_S^2 + 2\Re\mathrm{e}(Z_L) Z_S - 2Z_L Z_S)^2 - 4(-Z_S)(-Z_L - Z_L Z_S^2 + Z_S + \Re\mathrm{e}(Z_L)^2 Z_S + \Im\mathrm{m}(Z_L)^2 Z_S)}}{2(-Z_S)} \\
&= \frac{-(-1 - Z_S^2 + 2\Re\mathrm{e}(Z_L) Z_S - 2Z_L Z_S) \pm \sqrt{(-1 - Z_S^2 + 2\Re\mathrm{e}(Z_L) Z_S - 2Z_L Z_S)^2 - 4(-Z_S)(-Z_L - Z_L Z_S^2 + Z_S + \Re\mathrm{e}(Z_L)^2 Z_S + \Im\mathrm{m}(Z_L)^2 Z_S)}}{2(-Z_S)} \\
&= \frac{1 + Z_S^2 - 2\Re\mathrm{e}(Z_L) Z_S + 2Z_L Z_S \pm \sqrt{(-1 - Z_S^2 + 2\Re\mathrm{e}(Z_L) Z_S - 2Z_L Z_S)^2 + 4Z_S(-Z_L - Z_L Z_S^2 + Z_S + \Re\mathrm{e}(Z_L)^2 Z_S + \Im\mathrm{m}(Z_L)^2 Z_S)}}{-2Z_S} \\
&= \frac{Z_S^2 - 2\Re\mathrm{e}(Z_L) Z_S + 2Z_L Z_S + 1 \pm \sqrt{(-1 - Z_S^2 + 2\Re\mathrm{e}(Z_L) Z_S - 2Z_L Z_S)^2 + 4Z_S(-Z_L - Z_L Z_S^2 + Z_S + \Re\mathrm{e}(Z_L)^2 Z_S + \Im\mathrm{m}(Z_L)^2 Z_S)}}{-2Z_S} \\
&= \frac{Z_S^2 - 2\Re\mathrm{e}(Z_L) Z_S + 2Z_L Z_S + 1 \pm \sqrt{(-1 - Z_S^2 + 2\Re\mathrm{e}(Z_L) Z_S - 2Z_L Z_S)^2 + 4Z_S(-Z_L - Z_L Z_S^2 + Z_S + \Re\mathrm{e}(Z_L)^2 Z_S + \Im\mathrm{m}(Z_L)^2 Z_S)}}{-2Z_S} \\
&= \frac{Z_S^2 - 2\Re\mathrm{e}(Z_L) Z_S + 2Z_L Z_S + 1 \pm \sqrt{(-1 - Z_S^2 + 2\Re\mathrm{e}(Z_L) Z_S - 2Z_L Z_S)^2 - 4Z_L Z_S - 4Z_L Z_S^3 + 4Z_S^2 + 4\Re\mathrm{e}(Z_L)^2 Z_S^2 + 4\Im\mathrm{m}(Z_L)^2 Z_S^2}}{-2Z_S} \\
&= \frac{Z_S^2 - 2\Re\mathrm{e}(Z_L) Z_S + 2Z_L Z_S + 1 \pm \sqrt{Z_S^4 - 4\Re\mathrm{e}(Z_L) Z_S^3 + (8\Re\mathrm{e}(Z_L)^2 - 8\Re\mathrm{e}(Z_L) Z_L + 4Z_L^2 + 4\Im\mathrm{m}(Z_L)^2 + 6) Z_S^2 - 4\Re\mathrm{e}(Z_L) Z_S + 1}}{-2Z_S}
\end{aligned}
$$

CHAPTER 10

RF TRANSMISSION SYSTEMS

10.1 Transmission Line, Balun, & Antenna

In this demonstration we combine some of the basic concepts we already covered in a more complex example. Here we show an AC voltage source with source impedance Z_S connected to a transmission line of characteristic impedance of Z_0. The transmission line is then terminated with a balun, modeled as a transformer, and then an antenna with impedance Z_L.

The best way to approach the problem is to start at the load and find the thevenin equivalent impedance of each section, reducing it one piece at a time. We start by looking at the balun and antenna segment and finding the thevenin equivalent of this portion as seen in figure 12.

As in the past the first step is to define the mutual inductance.

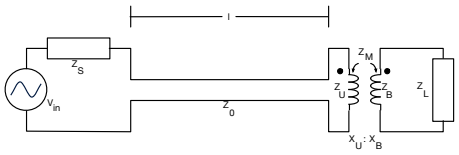

Figure 10.1: Transmission line, balun, and antenna system.

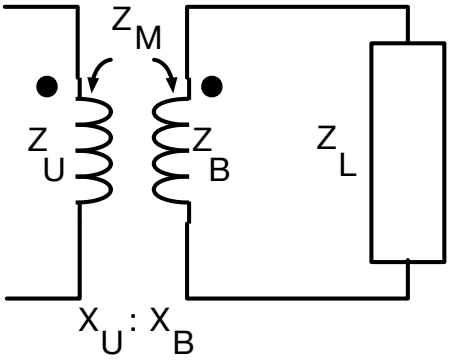

Figure 10.2: Balun and antenna portion of the system.

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